A pesky hypothesis for the integral test and the comparison tests:

$a_n \geq 0$: not all series have positive terms

Ex 1 Does $\sum_{n=1}^{\infty} \frac{-1}{n^2+1}$ converge or diverge?

(a) Converges

(b) Diverges

$\sum_{n=1}^{\infty} \frac{-1}{n^2+1} = -\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ ← converges

Conclusion: If all terms are < 0, factor out -1.

Ex 2 Can you apply either comparison test to $\sum_{n=1}^{\infty} \frac{n^2-7}{n^4+5}$?

$= \frac{-6}{6} + \frac{-3}{21} + \frac{2}{86} + \frac{9}{261} + \cdots$

Recall: only need $a_n \geq 0$ for $n \geq n_0$ for some n_0 to use comparison tests

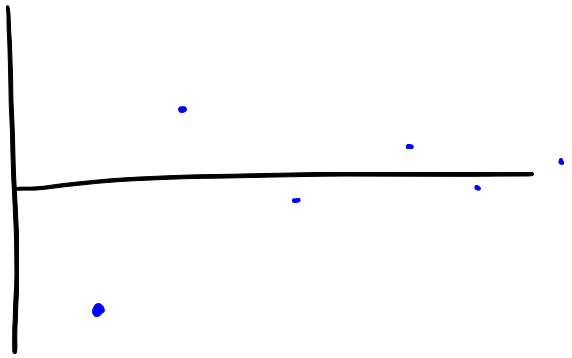
$\frac{n^2-7}{n^4+5} \geq 0$ for $n \geq 3$

$\frac{n^2-7}{n^4+5} \leq \frac{n^2}{n^4+5} \leq \frac{n^2}{n^4} = \frac{1}{n^2} \rightarrow \sum_{n=1}^{\infty} \frac{n^2-7}{n^4+5} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\sum \frac{1}{n^2}$ converges by p-test ($p=2$)

By D.C.T. $\sum_{n=1}^{\infty} \frac{n^2-7}{n^4+5}$ converges.

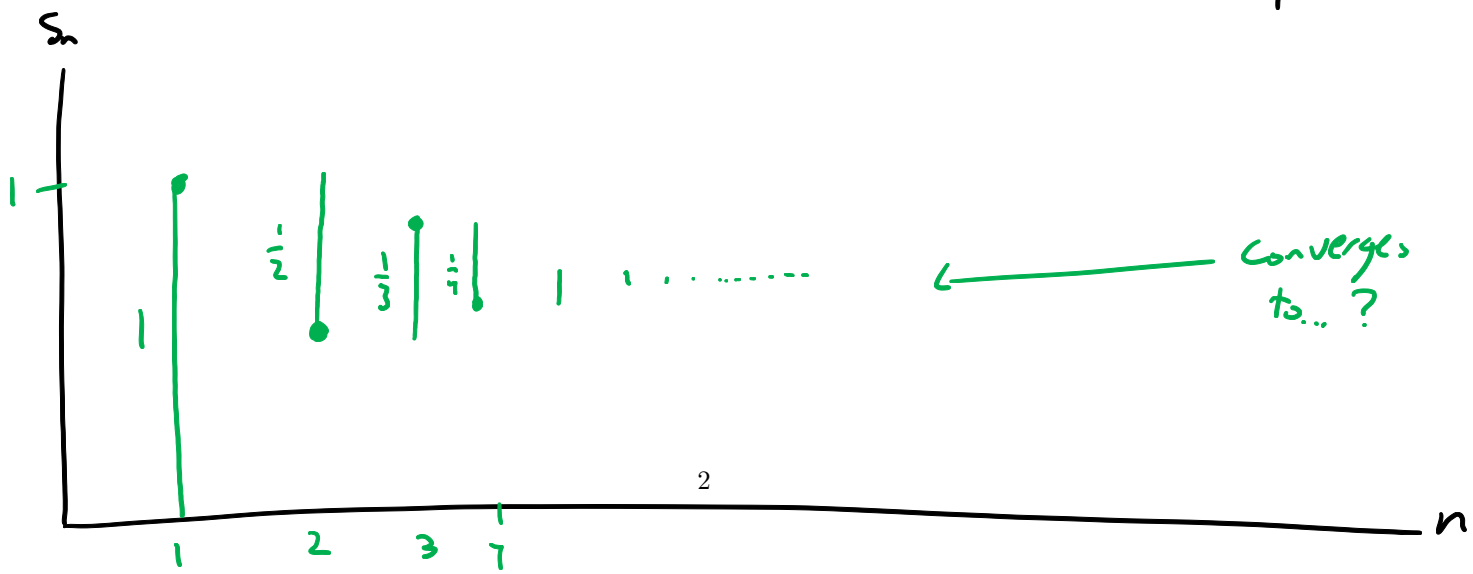
Ex 3 Can you apply the integral test or either comparison test to $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$? What are the partial sums of this series?



$\frac{(-1)^n}{n}$ is not eventually positive or eventually negative → no integral or comparison tests

Partial Sums

n	1	2	3	4	5	
$\frac{(-1)^{n+1}}{n}$	$\frac{(-1)^2}{1} = 1$	$\frac{(-1)^3}{2} = -\frac{1}{2}$	$\frac{(-1)^4}{3} = \frac{1}{3}$	$-\frac{1}{4}$	$\frac{1}{5}$	...
S_n	1	$1-\frac{1}{2} = \frac{1}{2}$	$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$	$\frac{5}{6} - \frac{1}{4} = \frac{7}{12}$	$\frac{7}{12} + \frac{1}{5} = \frac{47}{60}$	----



Def: An alternating series is any series of the form

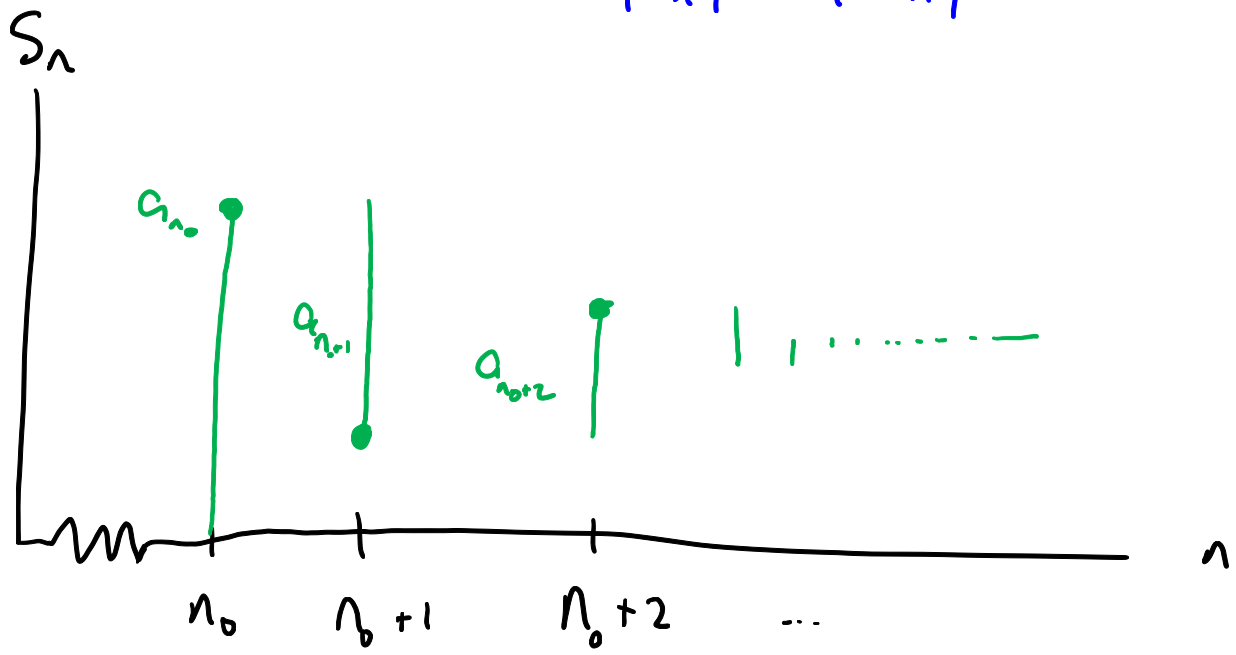
$$\sum_{n=n_0}^{\infty} (-1)^n b_n \quad \text{OR} \quad \sum_{n=n_0}^{\infty} (-1)^{n+1} b_n$$

where $b_n > 0 \quad \forall n \geq n_0$

Thm: (Alternating Series Test) If a series $\sum_{n=n_0}^{\infty} a_n$ is alternating AND the sequence $\{|a_n|\}_{n=n_0}^{\infty}$ converges monotonically to 0 then the series converges

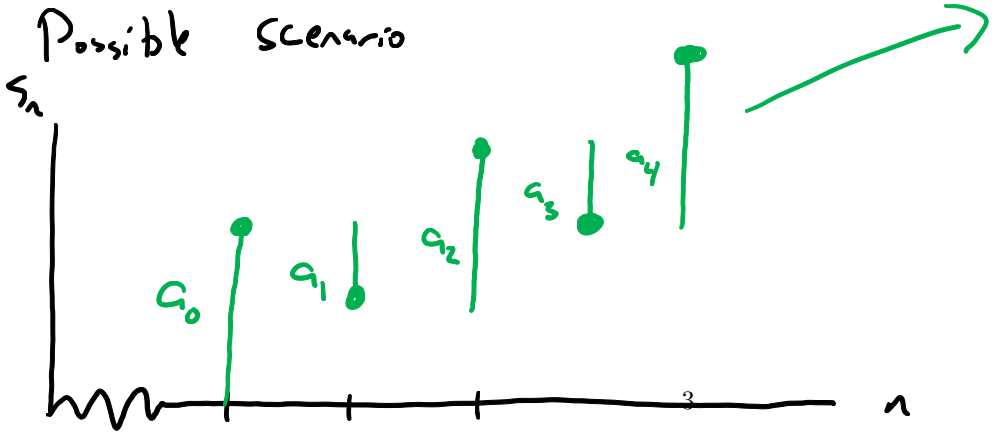
Q. Why is this true?

$|a_n| \geq |a_{n+1}|$



Q. What if not monotone?

Possible scenario



Ex 4 Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges.

$\frac{1}{n^2} > 0$, so $\frac{(-1)^{n+1}}{n^2}$ is alternating

$|a_n| = \left|\frac{(-1)^{n+1}}{n^2}\right| = \frac{1}{n^2} \Rightarrow \lim_{n\to\infty} |a_n| = 0$ ✓

$|a_{n+1}| = \left|\frac{(-1)^{n+2}}{(n+1)^2}\right| = \frac{1}{(n+1)^2}$

$\frac{1}{n^2} \geq \frac{1}{(n+1)^2}$ for all n, so $|a_n| \geq |a_{n+1}|$ ✓

By AST, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges

PQ

Ex 5 Can we use the alternating series test for $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ when $p \leq 0$?

$\frac{1}{n^p} \geq 0$, so $\frac{(-1)^{n+1}}{n^p}$ is alternating

$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n^p} = \lim_{n\to\infty} n^{(-p)} = \infty$ ← positive

$\Rightarrow$ A.S.T. does not apply

remainder

Q. Can we estimate the value of a convergent alternating series?

$$R_k = b_k - b_{k+1} + b_{k+2} - b_{k+3} + \cdots \qquad (b_n \to 0 \ \forall n)$$

$$= b_k - \underbrace{(b_{k+1} - b_{k+2})}_{\geq 0} - \underbrace{(b_{k+3} - b_{k+4})}_{\geq 0} - \underbrace{(\quad)}_{\geq 0} - \cdots$$

$$\leq b_k$$

Ex 6 Estimate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ with error $< 10^{-5}$.

Want $R_k < 10^{-5}$

Since $R_k \leq b_k$, we're good if $b_k < 10^{-5}$

$b_n = \frac{1}{k}$

$\frac{1}{k} < 10^{-5} \iff 10^5 < k$, so take $k = 10^5 + 1$

$$S_{10^5+1} = \sum_{n=1}^{10^5+1} \frac{(-1)^{n+1}}{n} \approx 0.6931$$

Def: A series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges. If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges we say $\sum_{n=1}^{\infty} a_n$ converges conditionally.

Thm: (Absolute Convergence Test) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof: Unenlightening

* Note: The converse is not true! $\sum_{n=1}^{\infty} a_n$ can converge without $\sum_{n=1}^{\infty} |a_n|$ converging (e.g. $a_n = \frac{(-1)^{n+1}}{n}$)

Ex 7 Show that $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges.

$$\left|\frac{\sin(n)}{n^2}\right| \simeq \frac{1}{n^2}$$

Note $0 \le \left|\frac{\sin(n)}{n^2}\right| \le \frac{1}{n^2}$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test ($p=2$)

By DCT, $\sum_{n=1}^{\infty} \left|\frac{\sin(n)}{n^2}\right|$ converges

By ACT $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges.

Note: Infinite sums are WEIRD.

- If $\sum_{n=1}^{\infty} a_n$ converges absolutely, you can rearrange the terms in the sum and the value doesn't change
- If $\sum_{n=1}^{\infty} a_n$ converges conditionally, rearranging terms can change the value

Fact:

Ex: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

$\sum_{n=1}^{\infty} \frac{1}{2n-1} \approx \sum_{n=1}^{\infty} \frac{1}{2n} = \infty$

Find k s.t. $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k-1} > 10$

Subtract $\frac{1}{2}$: $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k-1} - \frac{1}{2} > 9$

Add more terms with odd denoms. until

$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k-1} - \frac{1}{2} + \frac{1}{2(k+1)-1} + \cdots + \frac{1}{2j-1} > 100$

Subtract $\frac{1}{4}$

Etc. etc... Can rearrange so that sum diverges to ∞

Conditionally convergent series can be rearranged to converge to any value you want!